

The Miyazaka-Yau inequality and
the delta invariant
for Fano varieties

Masataka Iwai
Chenodars (the Univ. of Osaka)

Joint work with Tomoyuki Hisamoto
(Nagoya Univ.)
K-stability

Thm (Yau 78)

$$n = \dim X \geq 2$$

X : sm proj var/ $\mathbb{C}$, K_X ample

Then
$$\left[2(n+1) \underbrace{C_2(X)}_{C_2(T_X)} - n \underbrace{C_1(X)^2}_{C_1(T_X)^2 = C_1(-K_X)^2} \right] (K_X)^{n-2} \geq 0$$

(★)

Moreover, if the equality holds,
then $X_{\text{univ}} \underset{\text{biholo}}{\simeq} \{z \in \mathbb{C}^n \mid |z| \leq 1\}$

Rem $n=2 \Rightarrow$ (★) $\underbrace{3C_2(X) - C_1(X)^2}_{\geq 0}$

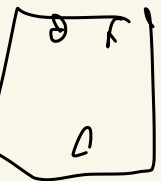
[Miyazaki 79]

(Miyazaki inequality)

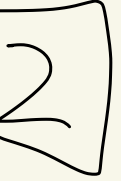


Miyazaki-Yau inequality

Why MY ineq hold?

  K_X ample $\Rightarrow \exists$ Kähler-Einstein metric
(Yau 78)

$$(\text{Ric}(w) = -w \in C_1(X))$$

 $\exists$ KE $\Rightarrow$ MY ineq holds
[Chen-Ogura 75]

Ques What if $-K_X$ is ample?
(ie Fano case??)

MY ineq holds or not??

$$\left[2(n+1)G_2(X) - nG_1(X)^2 \right] (-K_X)^{n-2} \geq 0??$$

Answer No MX neg. fails
for Fav var.

Ex $X = \text{Proj}(G \oplus G(3)) \rightarrow \mathbb{CP}^3$
Fav toric

$$n=4 \quad G_2(X) \cdot G_1(X)^2 = 296$$

$$G_1(X)^4 = 800$$

$$\begin{aligned} & \underline{2(n+1)G_2(X) - nG_1(X)^2} G_1(X)^2 \\ & = -240 < 0 \end{aligned}$$

Q, When does the Mordell-Weil
hold for Fano var.

A1 $\Rightarrow$ KE by Chen-Ogino

A2 delta invariant $\delta(X) \geq 1$

(cf Tian 92) Greb-Kebekus-Peternell 22

$\&$
Pruel-Grehanica-Păcur

What is the delta invariant?

$$\delta(X) (\geq 0)$$

(by Fujita-Otake 18)

Property

$$\delta(x) > 1 \stackrel{\text{BBJ}}{\Rightarrow} \neg KE \stackrel{\text{Berman?}}{\Rightarrow} \delta(x) \geq 1$$


Mixing.

Answer $\delta(x) \geq 1 \Rightarrow$ Mixing holds

Q. What if $f(x) < 1$??

Can we modify Myineq.
using delta invariant??
~

A. Yes Our Main result.

Thm [Hisamoto-I, 26] —
 X sm proj var / $\mathbb{C}$ $n = \dim X \geq 2$
 K_X is ample.

Then

$$\left[2(n+1) G_2(X) - n G_1(X)^2 \right] G_1(X)^2$$
$$\geq -n (1 - f'(x))^2 G_1(X)^n$$

Here $f'(x) = \min \{1, f(x)\}$

Moreover if equality holds,
then $X \simeq \mathbb{CP}^h$.

Ex $X = \mathbb{P}(G \oplus G(3)) \rightarrow \mathbb{CP}^3$

$$f'(x) = f(x) = \frac{500}{767} < 1$$

$$\begin{aligned} & \rightarrow h(1 - f'(x))^2 G(x)^h \\ & = - \frac{228124800}{588289} \approx \underline{-387}. \end{aligned}$$

$$[2(n+1)G_2(x) - hG(x)^2] G(x)^2 = \underline{-240}$$

Singular case?

X KLT var

We can define orbifold
2nd Chern class

$$\hat{c}_2(X) := \hat{c}_2(TX)$$

(by Greb-Kebekus-Peternell
-Tajiri 19)

Thm [Hisamoto-I. 26] (KLT var)

$X = \text{KLT var.}, n = \dim X \geq 2$

If $-K_X$ is ample,

$$\left[2(n+1)C_2(X) - nC_1(X)^2 \right] C_1(X)^{n-2} \\ \geq -n(1 - \delta'(X))^2 C_1(X)^n$$

$$\delta'(X) := \min \{1, \delta(X)\}$$

Moreover if equality holds,

then, $\exists X' \rightarrow X$

finite cover étale in codim 1

$$\underline{X' \simeq \mathbb{P}(1, \dots, 1, b) \quad b \geq 1}$$

Pf of main result (X sm, $\nrightarrow$ examp)

How to prove MX ineq
if $\hat{d}(X) \geq 1$.

Tian's idea.

MX in eq
||

Bogomolov - Gieseker ineq (1)
of

$rk \leq n+1$ vector bundle. (2)
with

$$c_2(V) = c_2(X) \otimes c_1(V) = c_1(X)$$

① Thm [Bogomolov-Gieseker ineq]

H : ample

F : H -slope semistable of
 ~~$\neq$~~ rank r .

$$\mu_H(F) = \frac{c_1(F)H^{n-1}}{rk F} \text{ slope,}$$

semistable, $\forall S \subset F \Rightarrow$

$$\mu_H(S) \leq \mu_H(F)$$

Then

$$[2(rk F)c_2(F) - (rk F - 1)c_1(F)^2]$$

$$\cdot H^{n-2} \geq 0.$$

② Tian's Canonical Extension sheaf

$$C_1(X) \in H^1(X, \Omega_X^1) \simeq \text{Ext}^1(T_X, G_X)$$

$$\leadsto 0 \rightarrow G_X \rightarrow V \rightarrow T_X \rightarrow 0$$

canonical ext sheaf

$$\left[\begin{array}{l} \text{rk } V = n+1, \\ \end{array} \right.$$

$$\left[\begin{array}{l} G_2(V) = G_2(T_X) = G_2(X) \\ \end{array} \right.$$

$$\left[\begin{array}{l} \text{rk } C_1(V) = C_1(T_X) = C_1(X) \\ \end{array} \right.$$

Thm [Tian92, Li20, DGP24]

$\delta(X) \geq 1$ $\Rightarrow V$ is K_X -slope
semistable

pf of MY ineq in $\delta(X) \geq 2$

$$\delta(X) \geq 2$$

$\Rightarrow V: -K_X$ slope semistable

Thm

$$\Rightarrow \left[2(hK_V)G_2(V) - (hK_V - 1)G_1(V)^2 \right] \cdot (-K_X)^{h-2} \geq 0$$

BGU

|| (2)

$$\left[2(h+1)G_2(X) - hG_1(X)^2 \right] (-K_X)^{h-1} \geq 0$$

How about $\delta(X) < 1$ case?

$\delta(X) \geq 1$ case,

Canonical
[a] Ext sheaf
— K_X semistable

+

[b] BG ineq

$\Downarrow$

MY ineq.

$\delta(X) < 1$ case

(a')

Bounds on
 $\mu^{\max}(V)$ & $\mu^{\min}(V)$

+

(b')

Langer's
ineq

$\Downarrow$

MY ineq
with determinant

(a) Thm [Tian 92, Xu-Zhuang 26
Smooth (with Aperiodic Gracia),
 Huisman-Ia 26]

V : canonical ext sheet

$$\begin{aligned} \underline{\mu^{\max}(V)} &= \sup_{\substack{-k_x \\ \text{subsheet}}} \{ \mu(F) \mid F \subset V \} \\ &\leq \left(1 - \delta'(X) \left(1 - \frac{1}{n+1} \right) \right) C_1(X)^n \end{aligned}$$

$$\begin{aligned} \underline{\mu^{\min}(V)} &= \inf \{ \mu_{-k_x}(Q) \mid V \twoheadrightarrow Q_{\text{torsion free}} \} \\ &\geq \delta'(X) \frac{1}{n+1} C_1(X)^n \end{aligned}$$

(b) Thm Langer's ineq

(Langer 04, Ir-Matsumura
Sm - Müller 25
Sing)

• H : ample

• F : torsion free sheaf of rk r .

$$\Rightarrow (2r c_2(F) - (r-1)c_1(F))H^{n-2}$$

$$\leq - \frac{r^2}{H^n} \left(\mu_{H}^{\max}(F) - \mu_{H}(F) \right)$$

$$\times \left(\mu_{H}(F) - \mu_{H}^{\min}(F) \right)$$

pf of Main result $f(X) \leq 2$

Ineq Substitute (a') into (b')

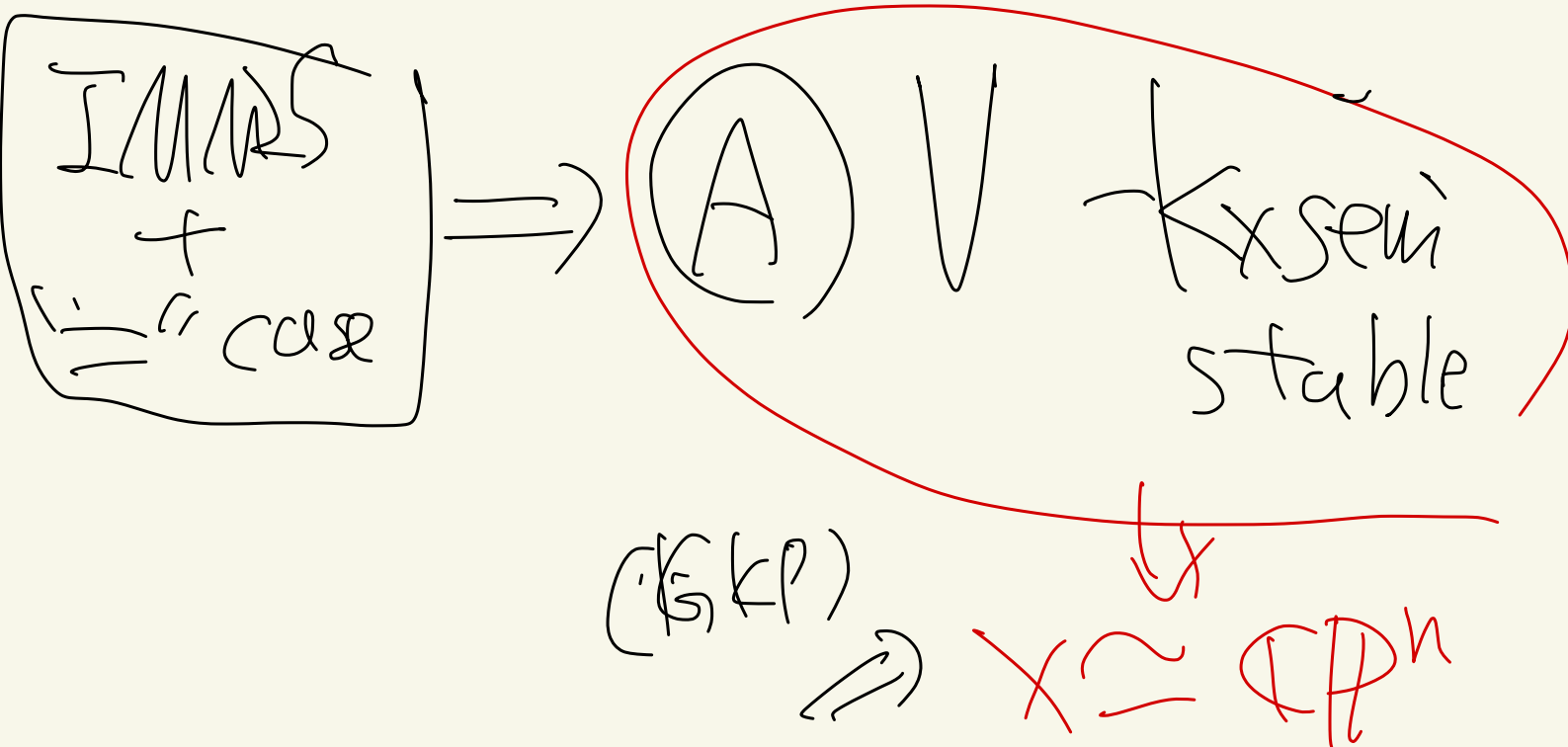
$$r = n+1, \quad C_2(V) = C_2(X)$$

$$C_1(V) = C_1(X)$$

Eg case (X smooth)

The equality of (b')

was studied in IMM25



$$(A) \Rightarrow \underline{\delta(X) \geq 1},$$

$$(2(n+1)C_2(V) - nC(V)^2)(C(X))^{n-2} = 0$$

or

(B) $\exists F, \mu$ line bundles,

$S \rightarrow X$

$$0 \rightarrow F \rightarrow V \rightarrow M^{\oplus n} \rightarrow 0$$

$$C_1(M) = \frac{\delta'(X)}{n+1} C_1(X)$$

$$\Rightarrow C_1(X) = \frac{n+1}{\delta'(X)} C_1(M)$$

$\Rightarrow \forall C \subset X$ curve.

$$K_X^* C = \frac{n+1}{\delta'(X)} \boxed{C_1(M) \cdot C}$$

$\underbrace{\delta'(X)}_{\text{vll}_{n+1}} \quad \underbrace{C_1(M) \cdot C}_{\text{vll}_2}$

$$\geq n+1$$

$$\Rightarrow X \simeq \mathbb{CP}^n$$

Cho-Miyazaki

- Shpihered Barman⁰⁰